Regularization of conical singularities in warped six-dimensional compactifications

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ABSTRACT: We study the regularization of the codimension-2 singularities in six-dimensional Einstein-Maxwell axisymmetric models with warping. These singularities are replaced by codimension-1 branes of a ring form, situated around the axis of symmetry. We assume that there is a brane scalar field with Goldstone dynamics, which is known to generate a brane energy momentum tensor of a particular structure necessary for the above regularization to be successful. We study these compactifications in both a non-supersymmetric and a supersymmetric setting. We see that in the non-supersymmetric case, there is a restriction to the admissible warpings and furthermore to the quantum numbers of the bulk gauge field and the brane scalar field. On the contrary, in the super-symmetric case, the warping can be arbitrary.

Contents

1. Introduction

There has been considerable attention the past years at brane models with codimension-2 branes in six-dimensional theories. This attention has mostly to do with an interesting property of their vacuum energy and has inspired the construction of models which may ameliorate the cosmological constant problem. The latter property is the fact that their vacuum energy instead of curving their world volume, merely changes the deficit angle in the geometry of the surrounding bulk [1]. Thus in principle, models which decouple the curvature of the brane from the brane vacuum energy can be constructed (selftuning models).

The most thoroughly discussed models of this kind were the ones with a bulk gauge field coupled to gravity [2] (for another possibility see [3, 4]). These models have generally a monopole solution which spontaneously compactifies the internal two-dimensional space [5]. This internal space can in principle have some conical defects which support codimension-2 branes. A usual assumption which we adopt also in the present paper is axial symmetry (see [6] for a more general case). This restricts the number of the codimension-2 branes to be at most two, situated at the north and south pole of the internal space. The size of spontaneously compactified internal space is classically determined in the case of a

purely Einstein-Maxwell system [5, 7], but behaves as a modulus in its supersymmetrized version [8 – 10], where a dilaton with appropriate coupling to the other fields is also present (for the stability of these models see [11, 12]). The selftuning property of these models is ruined however, by the flux quantization condition [13], unless one admits solutions with singularities more severe than conical [14].

The study of gravity on the codimension-2 branes is a difficult issue. Any simplistic way to discuss non-trivial brane geometries results to bulk singularities at the position of the brane much more serious than conical [15]. One way to confront this problem is to complicate the gravity dynamics by adding for example a Gauss-Bonnet term in the bulk or an induced gravity term on the brane [16]. This approach, however, leads to rather restrictive constraints on the matter content of the brane [17]. The most natural procedure is to consider a regularized version of the codimension-2 brane in Einstein gravity, where the brane acquires some thickness [18] in its transverse directions.

The latter approach has been followed in [19] in its simplest form, where the codimension-2 branes were replaced by thin ring-like codimension-1 branes warped around the axis of symmetry. The space close to the conical tip, which is cut by the codimension-1 brane, is replaced by appropriate bulk caps. This kind of regularization requires a specific form of the brane energy momentum tensor to work, which can be provided by a brane scalar field with Goldstone-like derivative couplings to the bulk gauge field. This Goldstone feature can be natural if this scalar field originates from a Higgs field whose radial component has been integrated out [20] at low enough energies. In [19] the regularization of the unwarped non-supersymmetric model had been provided and gravity on the codimension-1 brane was discussed.

In the present paper we generalize the above procedure by considering this type of regularization for codimension-2 brane models with general warping. Furthermore, we repeat the regularization also for a gauged supergravity model. We find that this regularization in the non-supersymmetric case cannot work for general warping (or equivalently for arbitrary monopole and brane scalar field quantum numbers). On the contrary, in the supersymmetric case no such constraint is present. In the next two sections we will present the solutions with the codimension-2 branes and then the regularization approach followed by the solutions with codimension-1 branes. At the end we will conclude and suggest a possible application of these regularized compactifications.

2. Non-supersymmetric warped compactifications

We discuss first the six-dimensional Einstein-Maxwell system which was originally found to spontaneously compactify an internal two-dimensional space [5]. We suppose that we have a six-dimensional gauge field $\mathcal{A}_M$ (with field strength $\mathcal{F}_{MN}$) coupled to gravity in the presence of a bulk cosmological constant Λ_0. The general axisymmetric solutions of such a model, involve compact spaces with sphere topology and two codimension-2 singularities at the poles of the deformed sphere [7]. The dynamics of this system can be encoded in

the following action

$$S = \int d^6x \sqrt{-g} \left(\frac{M^4}{2} R - \Lambda_0 - \frac{1}{4} \mathcal{F}^2 \right) - T_\pm \int d^4x \sqrt{-\gamma_\pm} \,, \tag{2.1}$$

where brane actions are added to the bulk part, with brane tensions $T_\pm$ for the codimension-2 branes situated at the north and south pole respectively. The tensor $\gamma_{\mu\nu}^\pm$ is the induced metric on the two 3-branes. The quantity M is the six-dimensional Planck mass.

The equations of motion in the bulk are easily obtained by variation of the metric and the gauge field and read

$$R_{MN} = \frac{1}{M^4} \left[\frac{\Lambda_0}{2} g_{MN} + \mathcal{F}_{MK} \mathcal{F}_N{}^K - \frac{1}{8} \mathcal{F}^2 g_{MN} \right] \,, \tag{2.2}$$

$$\partial_M \left(\sqrt{-g} \mathcal{F}^{MN} \right) = 0 \,. \tag{2.3}$$

In the following, we will first discuss the background solution with codimension-2 branes and then we will present the way to regularize these branes by lowering their codimension.

2.1 The general solution with codimension-2 branes

Let us recall the general bulk axisymmetric solution of the above system, presented in [7]. The solution can be obtained after a double Wick rotation from the one for the six-dimensional Reissner-Nordström black hole. Thus, the metric and the only non-zero component of the gauge field strength are

$$ds_6^2 = \rho^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\rho^2}{F} + c_0^2 F d\theta^2 \,, \tag{2.4}$$

$$\mathcal{F}_{\rho\theta} = -\frac{b_0}{\rho^4} \,, \tag{2.5}$$

$$\text{with} \quad M^4 F(\rho) = -\frac{\Lambda_0}{10} \rho^2 - \frac{b_0^2}{12 c_0^2 \rho^6} + \frac{\mu_0}{\rho^3} \,, \tag{2.6}$$

where c_0 is a constant which can be absorbed in the angular coordinate. The reason why we keep it in the metric will be explained later. Note that the dimensions of the various quantities in the solution are $[\Lambda_0] = [b_0^2] = [\mu_0] = [F^3] = M^6$, $[\rho] = [c_0] = M^0$, and $[\theta] = M^{-2}$.

The metric function $F(\rho)$ has generically many real roots. Without loss of generality, we will suppose that two of these roots are positive and we will consider the space $0 < \rho_- < \rho < \rho_+$. From the condition that $F(\rho_\pm) = 0$, we can express the parameters b_0 and μ_0 as a function of ρ_+ and $\alpha \equiv \rho_-/\rho_+$ [$0 < \alpha \leq 1$] as

$$b_0^2 = \frac{6}{5} c_0^2 \Lambda_0 \, \rho_+^8 \alpha^3 \frac{1 - \alpha^5}{1 - \alpha^3} \,, \tag{2.7}$$

$$\mu_0 = \frac{\Lambda_0}{10} \rho_+^5 \frac{1 - \alpha^8}{1 - \alpha^3} \,. \tag{2.8}$$

The parameter α expresses the degree of warping of the four-dimensional part of the metric as it will be evident shortly. The unwarped case corresponds to the value $\alpha = 1$,

when the two roots of the function F coincide. The latter limit is singular in the present gauge, and thus it would be difficult to compare results with the unwarped case. For this reason, we follow the prescription of [11] and we make the coordinate transformation

$$\rho = \rho_+ z \qquad \text{with} \quad z(r) = \frac{1}{2}[(1-\alpha)r + (1+\alpha)], \qquad (2.9)$$

$$\varphi = \frac{\Lambda_0}{M^4}\rho_+(1-\alpha)\theta . \qquad (2.10)$$

Then in the new coordinates the solution (2.4)-(2.6) becomes

$$ds_6^2 = z^2\rho_+^2\eta_{\mu\nu}dx^\mu dx^\nu + R_0^2\left[\frac{dr^2}{f} + c_0^2 f \, d\varphi^2\right], \qquad (2.11)$$

$$\mathcal{F}_{r\varphi} = -c_0 R_0 M^2 S \cdot \frac{1}{z^4}, \qquad (2.12)$$

$$\text{with} \quad f = \frac{4R_0^2}{\rho_+^2(1-\alpha)^2}F, \qquad (2.13)$$

$$\text{and} \quad S(\alpha) = \sqrt{\frac{3}{5}\alpha^3\frac{1-\alpha^5}{1-\alpha^3}}, \qquad (2.14)$$

with $R_0^2 = M^4/(2\Lambda_0)$ a quantity representing the average radius of the internal space. In the limit $\alpha \to 1$, we have that $S \to 1$. Note that $[f] = [\varphi] = M^0$. From the above definition we can verify that $f(r)$ depends only on the parameter α as

$$f(r) = \frac{1}{5(1-\alpha)^2}\left[-z^2 + \frac{1-\alpha^8}{1-\alpha^3} \cdot \frac{1}{z^3} - \alpha^3\frac{1-\alpha^5}{1-\alpha^3} \cdot \frac{1}{z^6}\right]. \qquad (2.15)$$

The range of the angular coordinates θ and φ have not been specified yet. We can in general take $\varphi \in [0, 2\pi\xi)$. The parameter ξ plays the same role as the parameter c_0 and by a coordinate transformation we can keep one of them and set the other to unity. In other words, only the product of ξ and c_0 has physical meaning. We will keep however both of them in the following with the purpose of making an easy comparison with the unwarped case, choosing them appropriately.

The two codimension-2 branes are situated at $\rho = \rho_\pm$, or in the new coordinates at $r(\rho_\pm) = \pm 1$ with $z(\rho_+) = 1$, $z(\rho_-) = \alpha$. Expanding the function f at $r \to \pm 1$ we get $f \to 2(1 \mp r)X_\pm$ with the quantities $X_\pm$ given by

$$X_+ = \frac{5 + 3\alpha^8 - 8\alpha^3}{20(1-\alpha)(1-\alpha^3)}, \qquad (2.16)$$

$$X_- = \frac{3 + 5\alpha^8 - 8\alpha^5}{20\alpha^4(1-\alpha)(1-\alpha^3)} . \qquad (2.17)$$

Then the metric around $r = \pm 1$ reads

$$ds_6^2 \approx z^2(\pm 1)\rho_+^2\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R_0^2}{X_\pm}\left[\frac{dr^2}{2(1\mp r)} + c_0^2 X_\pm^2 2(1\mp r)d\varphi^2\right]. \qquad (2.18)$$

The deficit angles at the two singularities are thus $\delta_\pm = 2\pi(1-\beta_\pm)$ with $\beta_\pm = c_0 X_\pm \xi$. These singularities are supported by codimension-2 branes with tensions related to the

deficit angles as $T_\pm = M^4 \delta_\pm$. Note that the quantities ξ and c_0 appear correctly together as stressed before.

In the limit $\alpha \to 1$, the warping of the space disappears and we have the case of a sphere with a deficit angle $\delta = 2\pi(1 - c_0\xi)$, since then $X_+ = X_- = 1$. In this case, the exact metric is rather simple because $f = (1 - r^2)$. Thus we get

$$ds_6^2 = \rho_+^2 \eta_{\mu\nu} dx^\mu dx^\nu + R_0^2 \left[\frac{dr^2}{(1 - r^2)} + c_0^2(1 - r^2)d\varphi^2 \right] , \qquad (2.19)$$

$$\mathcal{F}_{r\varphi} = -c_0 R_0 M^2 . \qquad (2.20)$$

As can be easily checked, the above metric coincides close to the branes with (2.18) for $\alpha \to 1$. With the coordinate transformation $r = \cos\omega$ we can obtain the more familiar form of the solution as in [2]

$$ds_6^2 = \rho_+^2 \eta_{\mu\nu} dx^\mu dx^\nu + R_0^2 \left[d\omega^2 + c_0^2 \sin^2\omega d\varphi^2 \right] , \qquad (2.21)$$

$$\mathcal{F}_{\omega\varphi} = c_0 R_0 M^2 \sin\omega . \qquad (2.22)$$

At this point we will partially fix ξ by postulating that it is a function of α and that in the limit of $\alpha \to 1$ it is $\xi(\alpha) \to 1$. With this limiting behaviour the deficit angle in the unwarped case is given by $\delta = 2\pi(1 - c_0)$. Thus, the quantity c_0 appearing in warped metric (2.4) corresponds to the deficit angle of the unwarped solution. But as we allow warping, the quantity $\xi(\alpha)$ is also in principle present in the range of the angular coordinate. Particular choices of $\xi(\alpha)$ (always with the above-mentioned limit) can make certain calculations easier.

The parameter ρ_+ can be absorbed to a x^μ-coordinate redefinition, so from now on we take $\rho_+ = 1$. The flux quantization of the above gauge field, gives us a quantization condition for the deficit angle. To find the quantization condition, we need the solution for the gauge field in two patches of the manifold (north and south) which correctly reproduce the flux when Stokes' theorem is applied. The gauge field in these patches should vanish at the poles since we have assumed that the branes carry no charge. These gauge field are given by

$$\mathcal{A}_\varphi^+ = c_0 R_0 M^2 \cdot \frac{2S}{3(1 - \alpha)} \left(\frac{1}{z^3} - 1 \right) , \qquad (2.23)$$

$$\mathcal{A}_\varphi^- = c_0 R_0 M^2 \cdot \frac{2S}{3(1 - \alpha)} \left(\frac{1}{z^3} - \frac{1}{\alpha^3} \right) , \qquad (2.24)$$

which in the unwarped case limit $\alpha \to 1$ gives

$$\mathcal{A}_\varphi^\pm = c_0 R_0 M^2(-r \pm 1) . \qquad (2.25)$$

Then single valuedness of the gauge transformation at the overlapping region, gives the following quantization condition

$$2c_0 R_0 M^2 \xi e\, Y = N \quad , \quad N \in \mathbb{Z}, \qquad (2.26)$$

$$\text{with} \quad Y = \frac{(1 - \alpha^3)}{3\alpha^3(1 - \alpha)} S , \qquad (2.27)$$

where e is a unit fundamental charge. In the limit when $\alpha \to 1$, it is $Y \to 1$ as expected.

We can choose the function $\xi(\alpha)$ to simplify the expression of the quantization condition, or the one of the quantities $\beta_\pm$. A choice $\xi = 1/Y$ is particularly helpful. In this case the quantization condition remains the same as in the case of no warping and the deficit angles are given by $\beta_\pm = c_0 X_\pm / Y$.

2.2 The regularization

We would like to regularize this model by substituting the codimension-2 branes by branes of lower codimension. This is possible, if we suppose that the space close to the codimension-2 singularities is cut at $r = r_\pm$ by codimension-1 branes and is then replaced by smooth caps, as done in [19] for the unwarped case. Let us denote by $\mathcal{M}_0$ the bulk between the codimension-1 branes, $\mathcal{M}_\pm$ the two caps and $\Sigma_\pm$ the two codimension-1 branes. The regularized space is drawn in figure 1. Then the action of the system is written as

$$S = \int_{\mathcal{M}_i} d^6x \sqrt{-g} \left(\frac{M^4}{2} R - \Lambda_i - \frac{1}{4}\mathcal{F}^2 \right) - \int_{\Sigma_\pm} d^5x \sqrt{-\gamma_\pm} \left(\lambda_\pm + \frac{v_\pm^2}{2}(\tilde{D}_{\hat{\mu}}\sigma_\pm)^2 + M^4\{\mathcal{K}\}_\pm \right),$$
(2.28)

where $\gamma_{\hat{\mu}\hat{\nu}}^\pm = g_{MN}\partial_{\hat{\mu}}X^M \partial_{\hat{\nu}}X^N$ is the induced metric on the branes, with $\hat{\mu} = \{\mu, \varphi\}$ a five dimensional index and X^M the bulk coordinates of the brane. [Here, for the static brane $X^N = (x^{\hat{\nu}}, r_\pm)$]. Also, Λ_i, $i = 0, \pm$ are the bulk cosmological constants in the three bulk regions and $\lambda_\pm$ are the tensions of the codimension-1 branes. Furthermore, the branes have Goldstone-like scalar fields $\sigma_\pm$ coupled to the bulk gauge field, through the combination $\tilde{D}_{\hat{\mu}}\sigma_\pm = \partial_{\hat{\mu}}\sigma_\pm - Ea_{\hat{\mu}}$, where $a_{\hat{\mu}} = \mathcal{A}_M \partial_{\hat{\mu}}X^M$ is the induced gauge field. Note that this in *not* the covariant derivative of $\sigma_\pm$ with respect to $a_{\hat{\mu}}$. [Since we will consider static branes, we have simply $a_{\hat{\mu}} = \mathcal{A}_{\hat{\mu}}$.] The last term in the brane action is the Gibbons-Hawking term for each brane. We denote $\{\mathcal{K}\} = \mathcal{K}^{\text{in}} + \mathcal{K}^{\text{out}}$ the sum of the extrinsic curvatures from each side of each brane. The extrinsic curvature is constructed using the normal to the brane n_M which points *inwards to the corresponding part of the bulk* each time (we use the conventions of [21]). For more details see the appendix B.

The scalar fields $\sigma_\pm$ originate from the phases of Higgs scalar fields, whose radial components have been integrated out for low enough energies. If C is the normalization of the Higgs phase $e^{iC\sigma_\pm}$, we cannot directly read it from our effective low energy brane action. We can fix this ambiguity by choosing, without loss of generality, $C = 1/\xi$ where ξ has to do with the range of the angular coordinate φ. This choice simplifies the solution for the scalar fields $\sigma_\pm$. As we present in detail in appendix A, the "charge" E is related to the charge of the parent Higgs field through ξ. If the Higgs field has unit fundamental charge e, the "charge" E is related to the latter as $E = \xi e$. It is important to note that the "charge" E need not be an integer multiple of the fundamental charge e. This does not contradict the well known fact that charges in compact spaces are quantized, since E is not a charge in the strict sense but just a parameter appearing in the effective brane action.

The equations of motion in the bulk are that same as (2.2), (2.3), where now we substitute Λ_i for the three bulk regions instead of Λ_0. The junction conditions on the
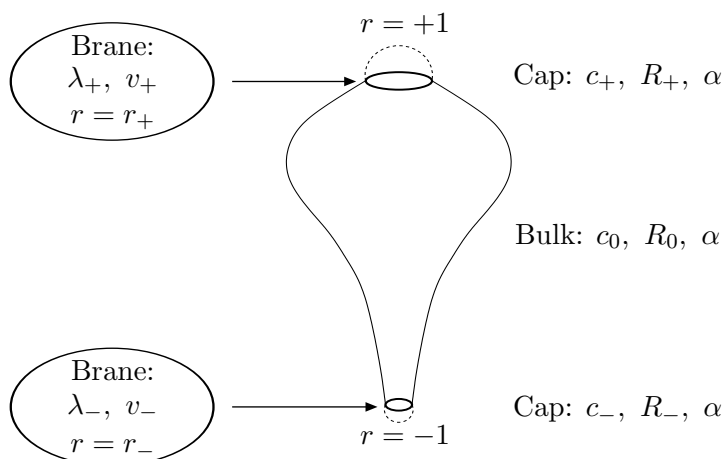
Figure 1: The internal space where the codimension-2 singularities have been regularized with the introduction of ring-like codimension-1 branes. As the positions of the rings tend to the poles i.e. when $r_{\pm} \to \pm 1$, we recover the conical brane limit. The parameters of the action and the solution are denoted in the appropriate part of the internal space.

branes are obtained by matching the surface terms from each side of the brane with the variation of the brane action and read

$$\{\hat{K}_{\hat{\mu}\hat{\nu}}\}_{\pm} = -\frac{1}{M^4}\left[-\lambda_{\pm}\gamma_{\hat{\mu}\hat{\nu}} + v_{\pm}^2\left(\tilde{D}_{\hat{\mu}}\sigma_{\pm}\tilde{D}_{\hat{\nu}}\sigma_{\pm} - \frac{1}{2}(\tilde{D}_{\hat{\kappa}}\sigma_{\pm})^2\gamma_{\hat{\mu}\hat{\nu}}\right)\right], \quad (2.29)$$

$$\{n_M\mathcal{F}^M_N\partial_{\hat{\kappa}}X^N\}_{\pm} = -Ev_{\pm}^2\tilde{D}_{\hat{\kappa}}\sigma_{\pm}, \quad (2.30)$$

where $\hat{K}_{\hat{\mu}\hat{\nu}} = K_{\hat{\mu}\hat{\nu}} - K\gamma_{\hat{\mu}\hat{\nu}}$, with $K_{\hat{\mu}\hat{\nu}} = \mathcal{K}_{MN}\partial_{\hat{\mu}}X^M\partial_{\hat{\nu}}X^N$. Finally, the equation of motion of the brane Goldstone fields is

$$\partial^{\hat{\mu}}\tilde{D}_{\hat{\mu}}\sigma_{\pm} = 0. \quad (2.31)$$

Before presenting the regularized solution, let us make a comment regarding the implementation of the above junction conditions. The metric in each side of the brane can be generally written as

$$ds_6^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{rr}dr^2 + g_{\varphi\varphi}d\varphi^2, \quad (2.32)$$

with $g_{\mu\nu}$, $g_{\varphi\varphi}$ continuous functions as they cross the brane. The derivatives of the latter metric components are of course discontinuous, which gives rise to the junction conditions. However, the metric components g_{rr} *need not* be continuous. We can always make a radial coordinate redefinition $r \to l(r)$ as in [19]

$$g_{ll}dl^2 = [g_{rr}^{\text{out}}\theta(r_c - r) + g_{rr}^{\text{in}}\theta(r - r_c)]dr^2, \quad (2.33)$$

with r_c the brane position, where g_{ll} is a continuous line element across the brane. The above junctions are usually understood for such a coordinate system. Then the normals on each side are opposite vectors $n_M^{\text{in}} = -n_M^{\text{out}}$. Nevertheless, the above equations still make sense for the case where $g_{rr}^{\text{in}} \neq g_{rr}^{\text{out}}$ at the position of the brane. In this case, the "normals" n_M in each direction are *not* opposite vectors (since they are normalized with different g_{rr}).

2.3 The general solution with codimension-1 branes

We can now present the background solution of the above regularized system. At the end, we will verify that taking the limit of $r_\pm \to \pm 1$, the solution of section 2.1 with the codimension-2 branes is correctly reproduced.

The solution in the bulk region $\mathcal{M}_0$ between the two branes ($r_- < r < r_+$), is the one described in section 2.1

$$ds_6^2 = z^2 \eta_{\mu\nu} dx^\mu dx^\nu + R_0^2 \left[\frac{dr^2}{f} + c_0^2 f \, d\varphi^2 \right] , \tag{2.34}$$

$$\mathcal{F}_{r\varphi} = -c_0 R_0 M^2 S \cdot \frac{1}{z^4} , \tag{2.35}$$

with $R_0^2 = M^4/(2\Lambda_0)$. On the other hand, the caps $\mathcal{M}_\pm$, with $r_+ < r < 1$ and $-1 < r < r_-$ respectively, are described by the solutions

$$ds_6^2 = z^2 \eta_{\mu\nu} dx^\mu dx^\nu + R_\pm^2 \left[\frac{dr^2}{f} + c_\pm^2 f d\varphi^2 \right] , \tag{2.36}$$

$$\mathcal{F}_{r\varphi} = -c_\pm R_\pm M^2 S \cdot \frac{1}{z^4} , \tag{2.37}$$

with $R_\pm^2 = M^4/(2\Lambda_\pm)$. Since the aim of the regularization is not to have codimension-2 branes present, one should demand that there is no deficit angle in the caps, i.e. that $\beta_\pm = c_\pm X_\pm \xi = 1$. Thus, we should fix the parameters $c_\pm$ as $c_\pm = 1/(X_\pm \xi)$, with $X_\pm$ as given in (2.16) and (2.17) respectively.

Continuity of the $g_{\varphi\varphi}$ metric component imposes the following relation between the curvatures in the three bulk regions and the parameters c_i

$$c_0 R_0 = c_\pm R_\pm . \tag{2.38}$$

Since the gauge field strength is continuous through the codimension-1 brane and taking into account the above relation, the solution for the gauge field which vanishes at the poles remains exactly the same as before (2.23), (2.24). Thus the quantization condition (2.26) is the same as in the codimension-2 model.

The solution of the Goldstone fields depends as discussed in the previous section, on the periodicity of φ and the normalization of $\sigma_\pm$ in the original Higgs theory. With the assumption we made in the previous section (i.e. $C = 1/\xi$ with C the normalization of $\sigma_\pm$) the Goldstone fields are simply

$$\sigma_\pm = n_\pm \varphi \quad , \quad \text{with} \quad n_\pm \in \mathbb{Z} . \tag{2.39}$$

The junction conditions will determine the brane parameters $\lambda_\pm$, $v_\pm$ and they will also give a relation between the quantum numbers of the brane scalar field $n_\pm$ and the bulk gauge field N. In appendix B we present all the necessary steps to arrive at the extrinsic curvatures $\hat{K}_{\hat\mu\hat\nu}$. Using these quantities we can compute the junction condition (2.29). Its $(\varphi\varphi)$ component reads

$$\pm 4M^4 \left. \frac{z'}{z} \sqrt{f} \right|_{r_\pm} \left(\frac{1}{R_0} - \frac{1}{R_\pm} \right) = \lambda_\pm - \frac{v_\pm^2 (n_\pm - E\mathcal{A}_\varphi^\pm)^2}{2c_0^2 R_0^2 f(r_\pm)} , \tag{2.40}$$

and its $(\mu\nu)$ component is given by

$$\pm M^4\sqrt{f}\left(3\frac{z'}{z}+\frac{1}{2}\frac{f'}{f}\right)\Bigg|_{r_\pm}\left(\frac{1}{R_0}-\frac{1}{R_\pm}\right)=\lambda_\pm+\frac{v_\pm^2(n_\pm-E\mathcal{A}_\varphi^\pm)^2}{2c_0^2R_0^2f(r_\pm)}\;. \tag{2.41}$$

Furthermore, the gauge field junction condition (2.30) can be easily evaluated as

$$\mp c_0 R_0 M^2 S\left.\frac{\sqrt{f}}{z^4}\right|_{r_\pm}\left(\frac{1}{R_0}-\frac{1}{R_\pm}\right)=Ev_\pm^2(n_\pm-E\mathcal{A}_\varphi^\pm)\;. \tag{2.42}$$

A relation between $n_\pm$ and N can be found by taking the difference of (2.41) and (2.40) and dividing by (2.42). Then, after substitution of $\mathcal{A}_\varphi$, c_0 from (2.23), (2.24), (2.26) and taking into account that $E=\xi e$, we obtain

$$n_\pm=\pm\frac{N}{2}w_\pm(\alpha)\quad\text{with}\quad w_\pm(\alpha)=\alpha^{2(1\mp1)}\frac{X_\pm}{YS}\;. \tag{2.43}$$

It can be easily seen that it is always $n_+-n_-=N$ because of the identity $X_++\alpha^4X_-=2YS$. In the limit $\alpha\to1$ we have $w_\pm\to1$, and thus $n_\pm=\pm N/2$ in agreement with [19]. It is worth noticing here that the above relations are independent of the parameter ξ.

Since the quantities $n_\pm$, N are integers, the above relation imposes a restriction to the values of the admissible warpings α. Simplifying w_+ as

$$w_+(\alpha)=\frac{2}{(1-\alpha^3)}\left[\frac{5(1-\alpha^8)}{8(1-\alpha^5)}-\alpha^3\right]\;, \tag{2.44}$$

shows that w_+ is bounded as $1<w_+(\alpha)<5/4$. Hence, we find that for $N>0$ the scalar quantum number is restricted as $N/2<n_+<5N/8$. [The constraint coming from n_- is complementary due to the relation $n_+-n_-=N$]. This excludes warped solutions for small monopole numbers $N\le4$. The first warped solution exists for $N=5$, with $n_+=3$, $n_-=-2$ and for warping $\alpha\approx.44$.

The above restriction dictates that not all warped solutions can be regularized in the way we have described in the previous section. A possible way to understand why, is that, due to the compactness of the space, there are topological constraints for the various fields (bulk gauge field and brane scalar field) which may not be possible to be satisfied simultaneously for any warping α. A similar conclusion would be reached if we had used a brane 3-form field $B_{\hat\mu\hat\nu\hat\kappa}$, instead of the scalar field σ, with a coupling $B_{(3)}\wedge A_{(1)}$, instead of $\sigma A_{(1)}$. However, the similarity of the conclusion would be due to the duality of the two dynamical systems in five dimensions. We are not aware of any other brane action which generates a brane energy momentum tensor of the anisotropic type that we should have for the particular regularization to work. Thus, we cannot conclude whether the above restriction is due to some fundamental physical obstruction, or it is an artifact of the specific regularization procedure which simply lowers the codimension. The later conjecture, however, is conceivable and it would certainly be interesting to investigate alternative brane actions which would clarify the generality of our result.

From the difference of (2.41) and (2.40) and dividing by the square of (2.42), we can obtain an expression for the parameters $v_\pm$ as

$$E^2 v_\pm^2 = \pm \left. \frac{3(1-\alpha)\sqrt{f}}{2z^4 \left(\frac{1}{z^3} - \frac{5(1-\alpha^8)}{8\alpha^3(1-\alpha^5)} \right)} \right|_{r_\pm} \left(\frac{1}{R_0} - \frac{1}{R_\pm} \right) , \tag{2.45}$$

which for the limit $\alpha \to 1$ reduces to $e^2 v_\pm^2 = \pm \frac{\sqrt{1-r_\pm^2}}{r_\pm c_0 R_0}(1 - c_0)$, in agreement with [19]. In the codimension-2 limit $r_\pm \to \pm 1$, the parameters $v_\pm$ vanish regardless of the warping.

Finally, taking the sum of (2.41) and (2.40) we obtain the values of the brane tensions

$$\lambda_\pm = \pm \frac{M^4}{20(1-\alpha)\sqrt{f}} \left(-8z + \frac{11}{2} \cdot \frac{1-\alpha^8}{1-\alpha^3} \cdot \frac{1}{z^4} - 4\alpha^3 \frac{1-\alpha^5}{1-\alpha^3} \cdot \frac{1}{z^7} \right)\bigg|_{r_\pm} \left(\frac{1}{R_0} - \frac{1}{R_\pm} \right) \tag{2.46}$$

which for the limit $\alpha \to 1$ reduces to $\lambda_\pm = \pm \frac{M^4 r_\pm}{2c_0 R_0 \sqrt{1-r_\pm^2}}(1 - c_0)$, in agreement with [19]. In the codimension-2 limit $r_\pm \to \pm 1$, the brane tensions $\lambda_\pm$ diverge.

The latter divergence is not worrisome when comparing with the codimension-2 limit, because the quantity with physical meaning is the total tension of the ring. This is obtained by integrating the four-dimensional part of the brane energy momentum tensor $t_{\mu\nu}^{(\pm)}$ over the azimuthal direction and reads

$$T_\pm = \int d\varphi \sqrt{g_{\varphi\varphi}} \, t_{00}^{(\pm)} = 2\pi\xi c_0 R_0 \sqrt{f(r_\pm)} \left[\lambda_\pm + \frac{v_\pm^2 (n_\pm - E\mathcal{A}_\varphi^\pm)^2}{2c_0^2 R_0^2 f(r_\pm)} \right] . \tag{2.47}$$

The divergent part of $\lambda_\pm$ is canceled by $\sqrt{f(r_\pm)}$. Then, evaluating the bracket from (2.41), we can take the codimension-2 limit $r_\pm \to \pm 1$. The tensions in this limit read

$$T_\pm = 2\pi M^4 \left(1 - c_0 X_\pm \xi \right) \tag{2.48}$$

and coincide with the tensions of the codimension-2 case presented in section 2.1.

3. Supersymmetric warped compactifications

In this section we will study the supersymmetrized version of the previous model. To do so, we consider the gauge supergravity model of Salam and Sezgin [8] where gravity is coupled to a six-dimensional gauge field $\mathcal{A}_M$, a Kalb-Ramond field $\mathcal{B}_{MN}$ and a dilaton field χ in a way that respects $\mathcal{N} = 2$ supersymmetry. The general axisymmetric solutions of such a model involve compact spaces with sphere topology and two codimension-2 singularities at the poles of the deformed sphere [9, 10, 20]. The bosonic action of the system (neglecting the Kalb-Ramond field which can be consistently set to zero background value) is given by

$$S = \int d^6 x \sqrt{-g} \left(\frac{M^4}{2} R - \frac{1}{4} e^{\chi/M^2} \mathcal{F}^2 - \frac{1}{2}(\partial_M \chi)^2 - 4g_0^2 M^4 e^{-\chi/M^2} \right) - T_\pm \int d^4 x \sqrt{-\gamma_\pm} , \tag{3.1}$$

where $T_\pm$ are the tension of the codimension-2 branes situated at the north and south pole respectively and g_0 is the gauge coupling of the gauged $\mathrm{U}(1)_R$ of the model. The metric $\gamma_{\mu\nu}^\pm$ is the induced metric on the two 3-branes and M the six-dimensional Planck mass.

In the absence of the brane terms, i.e. without codimension-2 singularities, the vacuum of solution of [8] respects $\mathcal{N} = 1$ supersymmetry. However, the brane terms in (3.1), necessary for warping to be allowed, break supersymmetry explicitly to $\mathcal{N} = 0$. In that respect we are not strict by calling the compactification supersymmetric. Nevertheless, we will keep this terminology since the solution stems from a gauged supergravity action with the addition of only localized terms which break supersymmetry.

The equations of motion in the bulk are easily obtained as

$$R_{MN} = 2g_0^2 e^{-\chi/M^2} g_{MN} + \frac{1}{M^4} \partial_M \chi \partial_N \chi$$
$$+ \frac{1}{M^4} e^{\chi/M^2} \left[\mathcal{F}_{MK} \mathcal{F}_N{}^K - \frac{1}{8} \mathcal{F}^2 g_{MN} \right],\tag{3.2}$$

$$\Box^{(6)} \chi = \frac{1}{4M^2} e^{\chi/M^2} \mathcal{F}^2 - 4g_0^2 M^2 e^{-\chi/M^2},\tag{3.3}$$

$$\partial_M \left(\sqrt{-g}\, e^{\chi/M^2} \mathcal{F}^{MN} \right) = 0.\tag{3.4}$$

The above equations of motion have the scaling symmetry

$$g_{MN} \to u\, g_{MN}\quad,\quad \chi \to \chi + M^2 \log u.\tag{3.5}$$

Note that this is not a symmetry of the action since $S \to u^2 S$. By applying this scaling symmetry to one of the solutions of the equations of motion, we can obtain a new (and inequivalent) solution.

In the following, we will first discuss the background solution with codimension-2 branes and then we will present the procedure to regularize these branes by lowering their codimension, in the same way we accomplished it in the non-supersymmetric case.

3.1 The general solution with codimension-2 branes

The general bulk axisymmetric solution of the above system, with a monopole in the internal space was provided in [9, 10]. The metric has, as in the non-supersymmetric case, a black hole form in the extra two dimensions. The warped solution reads explicitly

$$ds_6^2 = \rho\, \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dr^2}{F} + c_0^2 F d\theta^2,\tag{3.6}$$

$$\mathcal{F}_{\rho\theta} = -\frac{b_0}{\rho^3},\tag{3.7}$$

$$\chi = M^2 \log \rho,\tag{3.8}$$

$$\text{with}\quad M^4 F(\rho) = -2g_0^2 M^4 \rho - \frac{b_0^2}{4c_0^2 \rho^3} + \frac{\mu_0}{\rho}.\tag{3.9}$$

Observe that the above solution has the same structure as the non-supersymmetric solution (2.4)–(2.6) with the difference that ρ appears here with a lower power. This appears to be crucial in the following.

As before, we will consider the space $0 < \rho_- < \rho < \rho_+$ between two roots of $F(\rho)$. The parameters b_0 and μ_0 can be expressed as a function of ρ_+ and $\alpha \equiv \rho_-/\rho_+$ as

$$b_0 = \rho_+^2 \frac{M^2 c_0 \alpha}{R_0}, \tag{3.10}$$

$$\mu_0 = \rho_+^2 \frac{M^4(1+\alpha^2)}{4R_0^2} . \tag{3.11}$$

To make the comparison with the unwarped case more transparent, we will make the coordinate transformation

$$\rho = \rho_+ z \quad \text{with} \quad z(r) = \frac{1}{2}[(1-\alpha)r + (1+\alpha)], \tag{3.12}$$

$$\varphi = \frac{(1-\alpha)}{2R_0^2}\theta, \tag{3.13}$$

with $R_0^2 = 1/(8g_0^2)$ representing the average radius of the internal space. In the new coordinates the solution (3.6)-(3.9) becomes

$$ds_6^2 = \rho_+ \left\{ z\eta_{\mu\nu}dx^\mu dx^\nu + R_0^2 \left[\frac{dr^2}{f} + c_0^2 f d\varphi^2 \right] \right\}, \tag{3.14}$$

$$\mathcal{F}_{r\varphi} = -c_0 R_0 M^2 S \cdot \frac{1}{z^3}, \tag{3.15}$$

$$\chi = M^2 \log(\rho_+ z), \tag{3.16}$$

$$\text{with} \quad f = \frac{4R_0^2}{\rho_+(1-\alpha)^2}F, \tag{3.17}$$

$$\text{and} \quad S = \alpha . \tag{3.18}$$

From the above definition we can verify that $f(r)$ depends only on the parameter α as

$$f(r) = \frac{1}{(1-\alpha)^2} \left[-z - \frac{\alpha^2}{z^3} + \frac{(1+\alpha^2)}{z} \right] . \tag{3.19}$$

For the range of the coordinate φ we make the same assumption as before, i.e. there is $\varphi \in [0, 2\pi\xi)$, with ξ a function of α with the limit that as $\alpha \to 1$ it is $\xi(\alpha) \to 1$. The two codimension-2 branes are situated at $\rho = \rho_\pm$, or in the new coordinates at $r(\rho_\pm) = \pm 1$ with $z(\rho_+) = 1$, $z(\rho_-) = \alpha$. Expanding the function f at $r \to \pm 1$ we get $f \to 2(1 \mp r)X_\pm$ with the quantities $X_\pm$ given by

$$X_+ = \frac{1+\alpha}{2} \quad , \quad X_- = \frac{1+\alpha}{2\alpha^2} . \tag{3.20}$$

Then the metric around $r = \pm 1$ reads

$$ds_6^2 \approx \rho_+ \left\{ z(\pm 1)\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R_0^2}{X_\pm} \left[\frac{dr^2}{2(1 \mp r)} + c_0^2 X_\pm^2 2(1 \mp r)d\varphi^2 \right] \right\} . \tag{3.21}$$

The deficit angles at the two singularities are thus $\delta_\pm = 2\pi(1 - \beta_\pm)$ with $\beta_\pm = c_0 X_\pm \xi$ and are supported by codimension-2 branes with tensions $T_\pm = M^4 \delta_\pm$. For completeness, we present in appendix C the relation of the above gauge with the one used in [9].

The scaling symmetry (3.5) is manifested in the solution by the appearance of the quantity ρ_+. For the rest of the paper, we will choose $\rho_+ = 1$. We stress, however, that this is *not* a gauge choice as in the non-supersymmetric case, but a mere choice of a subset of solutions (from which all solutions can be obtained by applying the symmetry).

It is straightforward to compute the flux quantization condition of the gauge field. The gauge field in the northern and southern patches of the manifold are given as before by

$$\mathcal{A}_\varphi^+ = c_0 R_0 M^2 \frac{S}{(1-\alpha)} \left(\frac{1}{z^2} - 1 \right) , \tag{3.22}$$

$$\mathcal{A}_\varphi^- = c_0 R_0 M^2 \frac{S}{(1-\alpha)} \left(\frac{1}{z^2} - \frac{1}{\alpha^2} \right) , \tag{3.23}$$

which in the $\alpha \to 1$ limit gives the correct limit (2.25). Then single valuedness of the gauge transformation at the overlapping region, gives the quantization condition

$$2c_0 R_0 M^2 \xi e \, Y = N \quad , \quad N \in \mathbb{Z} , \tag{3.24}$$

$$\text{with} \quad Y = \frac{1+\alpha}{2\alpha^2} S . \tag{3.25}$$

A choice $\xi = 2\alpha/(1+\alpha)$ is particularly helpful, since then the quantization condition remains the same as in the case of no warping and the deficit angles are given by $\beta_+ = c_0 \alpha$ and $\beta_- = c_0/\alpha$.

3.2 The regularization

We will regularize the model in the same way that we did it for the non-supersymmetric case, by the introduction of codimension-1 branes. Using similar notation as before, the action of the system is written as

$$S = \int_{\mathcal{M}_i} d^6 x \sqrt{-g} \left(\frac{M^4}{2} R - \frac{1}{4} e^{\chi/M^2} \mathcal{F}^2 - \frac{1}{2} (\partial_M \chi)^2 - 4g_i^2 M^4 e^{-\chi/M^2} \right)$$
$$- \int_{\Sigma_\pm} d^5 x \sqrt{-\gamma_\pm} \left(V_\pm(\chi) + \frac{v_\pm^2}{2} (\tilde{D}_{\hat{\mu}} \sigma_\pm)^2 + M^4 \{\mathcal{K}\}_\pm \right) , \tag{3.26}$$

where g_i, $i = 0, \pm$ are the gauge couplings in the different bulk regions and $V_\pm(\chi)$ are the potentials for the dilaton on the codimension-1 branes. The internal space has the shape given in figure 1. The equations of motion in the bulk are the same as (3.2)-(3.4) and g_0 is substituted by g_i for the three bulk regions. The junction conditions on the branes are obtained by matching the surface terms from each side of the brane with the variation of the brane action and read

$$\{\hat{K}_{\hat{\mu}\hat{\nu}}\}_\pm = -\frac{1}{M^4} \left[-V_\pm \gamma_{\hat{\mu}\hat{\nu}} + v_\pm^2 \left(\tilde{D}_{\hat{\mu}} \sigma_\pm \tilde{D}_{\hat{\nu}} \sigma_\pm - \frac{1}{2} (\tilde{D}_{\hat{\kappa}} \sigma_\pm)^2 \gamma_{\hat{\mu}\hat{\nu}} \right) \right] \tag{3.27}$$

$$\{e^{\chi/M^2} n_M \mathcal{F}_N^M \partial_{\hat{\kappa}} X^N\}_\pm = -E v_\pm^2 \tilde{D}_{\hat{\kappa}} \sigma_\pm , \tag{3.28}$$

$$\{n_M \partial^M \chi\}_\pm = \frac{dV_\pm}{d\chi} . \tag{3.29}$$

3.3 The general solution with codimension-1 branes

The background solution of the above regularized system will be presented and we will show that taking the limit of $r_\pm \to \pm 1$, the solution of section 3.1 with the codimension-2 branes is correctly reproduced.

The solution in the bulk region $\mathcal{M}_0$ between the two branes ($r_- < r < r_+$) (see figure 1) is the one described in section 3.1

$$ds_6^2 = z\, \eta_{\mu\nu}dx^\mu dx^\nu + R_0^2 \left[\frac{dr^2}{f} + c_0^2 f d\varphi^2 \right], \tag{3.30}$$

$$\mathcal{F}_{r\varphi} = -c_0 R_0 M^2 S \cdot \frac{1}{z^3}, \tag{3.31}$$

$$\chi = M^2 \log z, \tag{3.32}$$

with $R_0^2 = 1/(8g_0^2)$. On the other hand the caps $\mathcal{M}_\pm$, with $r_+ < r < 1$ and $-1 < r < r_-$ respectively, are described by the solutions

$$ds_6^2 = z\, \eta_{\mu\nu}dx^\mu dx^\nu + R_\pm^2 \left[\frac{dr^2}{f} + c_\pm^2 f d\varphi^2 \right], \tag{3.33}$$

$$\mathcal{F}_{r\varphi} = -c_\pm R_\pm M^2 S \cdot \frac{1}{z^3} \tag{3.34}$$

$$\chi = M^2 \log z, \tag{3.35}$$

with $R_\pm^2 = M^4/(2\Lambda_\pm)$. In order that there is no deficit angle in the caps (i.e. when $\beta_\pm = c_\pm X_\pm \xi = 1$), we should demand that $c_\pm = 1/(X_\pm \xi)$, with $X_\pm$ as given in (3.20).

Continuity of the $g_{\varphi\varphi}$ metric component imposes again the relation

$$c_0 R_0 = c_\pm R_\pm. \tag{3.36}$$

Since the gauge field strength is continuous through the codimension-1 brane and taking into account the above relation, the solution for the gauge field which vanishes at the poles remains exactly the same as before (3.22), (3.23). Thus the quantization condition (3.24) is the same as in the codimension-2 model.

The solution of the Goldstone fields, with the normalization discussed in appendix A, is given as before

$$\sigma_\pm = n_\pm \varphi \quad, \quad \text{with} \quad n_\pm \in \mathbb{Z}. \tag{3.37}$$

The brane parameters $v_\pm$ and the value and the slope of the dilaton potential $V(\chi)$ (which implicitly depends on branes positions) will be provided by the junction conditions. Furthermore, we will obtain a relation between the quantum numbers of the brane scalar field $n_\pm$ and the bulk gauge field N. In appendix B, we present all the necessary steps to arrive at the extrinsic curvatures $\hat{K}_{\hat{\mu}\hat{\nu}}$. Using these extrinsic curvatures, the $(\varphi\varphi)$ component of the junction condition (3.27) reads

$$\pm 2M^4 \left. \frac{z'}{z}\sqrt{f}\right|_{r_\pm} \left(\frac{1}{R_0} - \frac{1}{R_\pm} \right) = V_\pm - \frac{v_\pm^2(n_\pm - E\mathcal{A}_\varphi^\pm)^2}{2c_0^2 R_0^2 f(r_\pm)}, \tag{3.38}$$

and its $(\mu\nu)$ component is given by

$$\pm \frac{M^4}{2} \sqrt{f} \left(3\frac{z'}{z} + \frac{f'}{f} \right)\bigg|_{r_\pm} \left(\frac{1}{R_0} - \frac{1}{R_\pm} \right) = V_\pm + \frac{v_\pm^2 (n_\pm - E\mathcal{A}_\varphi^\pm)^2}{2c_0^2 R_0^2 f(r_\pm)} . \tag{3.39}$$

Furthermore, the gauge field junction condition (3.28) can be easily evaluated as

$$\mp c_0 R_0 M^2 \, S \, \frac{\sqrt{f}}{z^2}\bigg|_{r_\pm} \left(\frac{1}{R_0} - \frac{1}{R_\pm} \right) = E v_\pm^2 (n_\pm - E\mathcal{A}_\varphi^\pm) , \tag{3.40}$$

and the scalar field junction condition (3.29) gives

$$\mp M^2 \frac{z'}{z} \sqrt{f}\bigg|_{r_\pm} \left(\frac{1}{R_0} - \frac{1}{R_\pm} \right) = \frac{dV_\pm}{d\chi} . \tag{3.41}$$

In the above equations we have denoted $V_\pm = V(\chi(r_\pm))$ and $dV_\pm/d\chi = (dV/d\chi)(\chi(r_\pm))$.

A relation between $n_\pm$ and N can be found by taking the difference of (3.39) and (3.38) and dividing by (3.40). Then, after substitution of $\mathcal{A}_\varphi$, c_0 from (3.22), (3.23), (3.24) and taking into account that $E = \xi e$, we obtain the simple result

$$n_\pm = \pm \frac{N}{2} . \tag{3.42}$$

From the above relation we see that the warping α is *not* restricted as in the non-supersymmetric case. Furthermore, the Goldstone field quantum number is related to the gauge field quantum number in exactly the same way as in the unwarped case. However, as we noted in the non-supersymmetric case, we cannot be sure if this conclusion depends on the particular regularization, before we try some alternative brane action.

From the difference of (3.39) and (3.38) and dividing by the square of (3.40), we can obtain an expression for the parameters $v_\pm$ as

$$E^2 v_\pm^2 = \pm \frac{2\alpha^2 (1-\alpha)\sqrt{f}}{(2\alpha^2 - (1+\alpha^2)z^2)}\bigg|_{r_\pm} \left(\frac{1}{R_0} - \frac{1}{R_\pm} \right) , \tag{3.43}$$

which for the limit $\alpha \to 1$ reduces to $e^2 v_\pm^2 = \pm \frac{\sqrt{1-r_\pm^2}}{r_\pm c_0 R_0}(1-c_0)$. In the codimension-2 limit $r_\pm \to \pm 1$, the parameters $v_\pm$ vanish regardless of the warping.

Taking the sum of (3.39) and (3.38) we obtain the values of the dilaton potential on the rings

$$V_\pm = \pm \frac{M^4(1-\alpha)}{4} \, \sqrt{f} \, \frac{(4z^4 - 3(1+\alpha^2)z^2 + 2\alpha^2)}{z(\alpha^2 - z^2)(1-z^2)}\bigg|_{r_\pm} \left(\frac{1}{R_0} - \frac{1}{R_\pm} \right) , \tag{3.44}$$

which for the limit $\alpha \to 1$ reduces to $V_\pm = \pm \frac{M^4 r_\pm}{2c_0 R_0 \sqrt{1-r_\pm^2}}(1-c_0)$. In the codimension-2 limit $r_\pm \to \pm 1$, the above values $V_\pm$ diverge. On the other hand (3.41) provides the values of the slope of the dilaton potential on the rings. In the limit $\alpha \to 1$ these slopes vanish.

As we had explained in the non-supersymmetric case, the divergence of $V_\pm$ in the codimension-2 limit is not worrisome. The total tension of the ring is obtained by integrating the four-dimensional part of the brane energy momentum tensor $t_{\mu\nu}^{(\pm)}$ over the azimuthal direction. This is given by

$$T_\pm = \int d\varphi \sqrt{g_{\varphi\varphi}}\, t_{00}^{(\pm)} = 2\pi\xi c_0 R_0 \sqrt{f(r_\pm)} \left[V_\pm + \frac{v_\pm^2 (n_\pm - E\mathcal{A}_\varphi^\pm)^2}{2c_0^2 R_0^2 f(r_\pm)} \right] . \qquad (3.45)$$

Then, evaluating the bracket in the above relation using (3.39) and taking the codimension-2 limit $r_\pm \to \pm 1$, we can find the tensions in this limit

$$T_\pm = 2\pi M^4 \left(1 - c_0 X_\pm \xi\right) , \qquad (3.46)$$

which coincide with the tensions of the codimension-2 case presented in section 3.1.

4. Conclusions

In the present work, we discussed the regularization of the codimension-2 singularities in Einstein-Maxwell systems with a monopole in the internal space. We have applied the regularization procedure of [19], which lowers the codimension of the branes, in axisymmetric backgrounds with general warping. In this way, instead of working with codimension-2 branes, which have rather restrictive gravitational dynamics, one can work with ring-like codimension-1 branes which are much better understood.

An important ingredient of the regularization was the Goldstone-like dynamics of a brane scalar field coupled to the bulk gauge field. We have shown that the above procedure works for any warped solution in the supersymmetric case, when the quantum number of the brane scalar field and the bulk gauge field are related as in (3.42). On the contrary, in the non-supersymmetric case, not all warpings are consistent with the corresponding quantum number constraint (2.43). This may, however, be an artifact of the scalar field dynamics which have been used to generate the necessary energy momentum tensor needed for the regularization to work.

An important application of the above regularizations would be the study of cosmology on the codimension-1 branes and its limit as the radius of these branes goes to zero (codimension-2 limit). This will shed light on the cosmological properties of the codimension-2 branes [15, 18]. A particular case, is the one in which the ring-like brane moves in the static bulk background and induces a mirage cosmological evolution on it, as e.g. in [22]. We plan to address this issue in a forthcoming publication [23].

A. Origin of the Goldstone field action

In this appendix we will discuss the origin of the brane scalar field action and justify the comments we made in the main text. Let us consider a complex Higgs field H coupled to the gauge field $\mathcal{A}_{\hat{\mu}}$. We know that since we have a compact dimension, every field living in it that couples to $\mathcal{A}_{\hat{\mu}}$, should have a charge which is an integer multiple of a fundamental charge e. For convenience we will assume that the Higgs' charge is equal to the fundamental charge e (although it could in principle be an integer multiple). The Higgs kinetic Lagrangian reads

$$-(D_{\hat{\mu}}H)(D_{\hat{\nu}}H)^*\gamma^{\hat{\mu}\hat{\nu}}\,, \tag{A.1}$$

with $D_{\hat{\mu}}H = \partial_{\hat{\mu}}H - ie\mathcal{A}_{\hat{\mu}}H$ the covariant derivative with respect to the gauge field. At low enough energies we can integrate out the massive radial part and focus only on the Goldstone mode of the Higgs. The latter is given by $H = \mathrm{v}\, e^{i\Sigma}$ and thus the Lagrangian becomes

$$-\mathrm{v}^2(\bar{D}_{\hat{\mu}}\Sigma)(\bar{D}_{\hat{\nu}}\Sigma)\gamma^{\hat{\mu}\hat{\nu}}\,, \tag{A.2}$$

with $\bar{D}_{\hat{\mu}}\Sigma = \partial_{\hat{\mu}}\Sigma - e\mathcal{A}_{\hat{\mu}}$, not to be confused as covariant derivative but to be just a convenient abbreviation. If we use the angular coordinate $\varphi \in [0, 2\pi\xi)$ as in the main text, the solution for the field Σ which respects the periodicity of the space is $\Sigma = n\varphi/\xi$ with $n \in \mathbb{Z}$. Had we used an angular coordinate $\Phi \in [0, 2\pi)$, the solution would be $\Sigma = n\Phi$. We see that the periodicity of the space directly enters in the solution of the brane scalar field.

However, we do not know in principle from the effective brane Lagrangian the normalization of the Goldstone field. Let us for example rescale the Goldstone field as $\Sigma = C\sigma$. Then the Higgs field is written as $H = \mathrm{v}\, e^{iC\sigma}$ and the Lagrangian reads

$$-\mathrm{v}^2(\bar{D}_{\hat{\mu}}[C\sigma])(\bar{D}_{\hat{\nu}}[C\sigma])\gamma^{\hat{\mu}\hat{\nu}} = -v^2(\tilde{D}_{\hat{\mu}}\sigma)(\tilde{D}_{\hat{\nu}}\sigma)\gamma^{\hat{\mu}\hat{\nu}}\,, \tag{A.3}$$

with $v = C\mathrm{v}$, $\tilde{D}_{\hat{\mu}}\sigma = \partial_{\hat{\mu}}\sigma - E\mathcal{A}_{\hat{\mu}}$ and $E = e/C$ a redefined "charge". The latter brane Lagrangian was used in the main text. The solution for the field σ is then $\sigma = \Sigma/C = n\varphi/(C\xi)$ with $n \in \mathbb{Z}$.

In the above presentation, there are two ambiguities: the normalization C of the Goldstone field (i.e. if it is σ or Σ that appears in the action) and the parameter ξ which has to do with the range of the angular coordinate. We can fix part of this ambiguity by imposing that $C = 1/\xi$. The solution for the field σ (which respects the periodicity of the space) is then $\sigma = n\varphi$ with $n \in \mathbb{Z}$, independent of the quantity ξ. The Higgs vev is then related to the parameter v appearing in the action as $v = \mathrm{v}/\xi$ and the "charge" E is given by $E = \xi e$. Let us note that the remaining ambiguity (that of the determination of ξ) can be further partially fixed by comparison with the unwarped solution as described in the main text.

We should stress here that the "charge" E of σ need not be a integer multiple of the fundamental charge e. This is because it is only the charge of the parent Higgs field appearing in the original Higgs Lagrangian that is subject to quantization. The fact that there is a non-integer "charge" for σ is merely due to the normalization of the Goldstone mode and the range of the angular variable.

B. Extrinsic curvatures

In this appendix we will calculate the extrinsic curvatures on the brane positions, which will be used in the main text to evaluate the junction conditions. Let the brane position in the bulk be $X^M(x^{\hat{\mu}})$, from which we evaluate the induced metric on the brane as $\gamma_{\hat{\mu}\hat{\nu}} = g_{MN}\partial_{\hat{\mu}}X^M\partial_{\hat{\nu}}X^N$. Firstly, we should calculate the normal vector of the brane n_M, which is orthogonal to all the tangent vectors of the brane $\partial_{\hat{\nu}}X^M$, that is $\partial_{\hat{\nu}}X^M n_M = 0$. The tangent vector in the brane time direction $u^M \propto \partial_0 X^M$ normalized to unit norm $u^M u^N g_{MN} = -1$ is known as the proper velocity of the brane. The normal vector is normalized as $n_M n_N g^{MN} = 1$.

Once the normal vector is computed, one can evaluate the projection tensor $h_{MN} = g_{MN} - n_M n_N$, which is related to the induced metric as $h^{MN} = \gamma^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}X^M\partial_{\hat{\nu}}X^N$ and satisfies also $\gamma_{\hat{\mu}\hat{\nu}} = h_{MN}\partial_{\hat{\mu}}X^M\partial_{\hat{\nu}}X^N$ due to the orthogonality of the normal to the tangent vectors. Afterwards, the extrinsic curvature is given by $\mathcal{K}_{MN} = h_M^K h_N^\Lambda \nabla_K n_\Lambda$ (the covariant derivative computed with g_{MN}) and with trace $\mathcal{K} = g^{MN}\mathcal{K}_{MN}$. The pullback of the extrinsic curvature on the brane is given by $K_{\hat{\mu}\hat{\nu}} = \mathcal{K}_{MN}\partial_{\hat{\mu}}X^M\partial_{\hat{\nu}}X^N$ and has the property that $K = \gamma^{\hat{\mu}\hat{\nu}}K_{\hat{\mu}\hat{\nu}} = \mathcal{K}$. The combination which appears in the junction conditions is $\hat{K}_{\hat{\mu}\hat{\nu}} = K_{\hat{\mu}\hat{\nu}} - K\gamma_{\hat{\mu}\hat{\nu}}$.

In our examples, since the branes are static, their velocity vectors are $u^M \propto (1, \vec{0}, 0, 0)$. The normal vectors of the branes with the above mentioned normalization are given then by $n_M = \sqrt{g_{rr}}(0, \vec{0}, 1, 0)$. Since the (rr) component of the metric in the gauge that we work is discontinuous, we have different normal vectors pointing *inwards* the cap n_M^{in} and the *outwards* to the bulk n_M^{out}. Their non-vanishing components, for the upper and the lower brane are respectively

$$n_r^{\text{in}} = \pm\frac{R_\pm}{\sqrt{f(r_\pm)}} \quad , \quad n_r^{\text{out}} = \mp\frac{R_0}{\sqrt{f(r_\pm)}} \ . \tag{B.1}$$

The projection tensors $h_{MN} = g_{MN} - n_M n_N$ on the two branes are then $h_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}}$ and $h_{rr} = h_{r\hat{\mu}} = 0$. Let us now split the presentation for the non-supersymmetric and the supersymmetric case.

In the **non-supersymmetric case**, the induced metrics $\gamma_{\hat{\mu}\hat{\nu}}^\pm$ on the branes are given simply by

$$ds_{5(\pm)}^2 = z^2(r_\pm)\eta_{\mu\nu}dx^\mu dx^\nu + R_\pm^2 c_\pm^2 f(r_\pm)d\varphi^2 \ . \tag{B.2}$$

The non-zero extrinsic curvature components then read $K_{\mu\nu} = -\Gamma_{\mu\nu}^r n_r$ and $K_{\varphi\varphi} = -\Gamma_{\varphi\varphi}^r n_r$. The six-dimensional Christoffel symbols that we need are

$$\Gamma_{\mu\nu}^r = -\frac{zz'f}{R_0^2}\eta_{\mu\nu} \quad , \quad \Gamma_{\varphi\varphi}^r = -\frac{1}{2}c_0^2 ff', \tag{B.3}$$

with $' \equiv d/dr$. With the above, we compute the extrinsic curvatures $\hat{K}_{\hat{\mu}\hat{\nu}}$. Inwards to the caps, for the upper and the lower brane, they are respectively

$$\hat{K}_{\mu\nu}^{\text{in}} = \mp\frac{\eta_{\mu\nu}}{R_\pm}z^2\sqrt{f}\left(3\frac{z'}{z} + \frac{1}{2}\frac{f'}{f}\right)\Big|_{r_\pm} \quad , \quad \hat{K}_{\varphi\varphi}^{\text{in}} = \mp 4c_\pm^2 R_\pm\frac{z'}{z}f^{3/2}\Big|_{r_\pm} \ . \tag{B.4}$$

Outwards to the bulk, for the upper and the lower brane, they are respectively

$$\hat{K}_{\mu\nu}^{\text{out}} = \pm \frac{\eta_{\mu\nu}}{R_0} z^2 \sqrt{f} \left(3\frac{z'}{z} + \frac{1}{2}\frac{f'}{f} \right)\Big|_{r_\pm} \quad , \qquad \hat{K}_{\varphi\varphi}^{\text{out}} = \pm 4c_0^2 R_0 \frac{z'}{z} f^{3/2}\Big|_{r_\pm} \, . \tag{B.5}$$

In the **supersymmetric case**, the induced metrics $\gamma_{\hat\mu\hat\nu}^\pm$ on the branes are given by

$$ds_{5(\pm)}^2 = z(r_\pm)\eta_{\mu\nu}dx^\mu dx^\nu + R_\pm^2 c_\pm^2 f(r_\pm)d\varphi^2 \, . \tag{B.6}$$

The six-dimensional Christoffel symbols that we need are

$$\Gamma_{\mu\nu}^r = -\frac{z'f}{2R_0^2}\eta_{\mu\nu} \quad , \quad \Gamma_{\varphi\varphi}^r = -\frac{1}{2}c_0^2 ff', \tag{B.7}$$

With the above, we compute the extrinsic curvatures $\hat{K}_{\hat\mu\hat\nu}$. Inwards to the caps, for the upper and the lower brane, they are respectively

$$\hat{K}_{\mu\nu}^{\text{in}} = \mp \frac{\eta_{\mu\nu}}{2R_\pm} z\sqrt{f}\left(3\frac{z'}{z} + \frac{f'}{f} \right)\Big|_{r_\pm} \quad , \qquad \hat{K}_{\varphi\varphi}^{\text{in}} = \mp 2c_\pm^2 R_\pm \frac{z'}{z} f^{3/2}\Big|_{r_\pm} \, . \tag{B.8}$$

Outwards to the bulk, for the upper and the lower brane, they are respectively

$$\hat{K}_{\mu\nu}^{\text{out}} = \pm \frac{\eta_{\mu\nu}}{2R_0} z\sqrt{f}\left(3\frac{z'}{z} + \frac{f'}{f} \right)\Big|_{r_\pm} \quad , \qquad \hat{K}_{\varphi\varphi}^{\text{out}} = \pm 2c_0^2 R_0 \frac{z'}{z} f^{3/2}\Big|_{r_\pm} \, . \tag{B.9}$$

With the above it is straightforward to evaluate the junction conditions (2.29), (2.30) in the non-supersymmetric case and the junctions (3.27), (3.28), (3.29) in the supersymmetric case.

C. An alternative gauge in the supersymmetric case

In this appendix we will present the relation of the gauge that we used for the solution in the supersymmetric case with the one used in [9] (slightly rescaled). Let us make the coordinate transformation

$$z(r) = W(R)^2 \, , \tag{C.1}$$

$$\psi = \frac{1+\alpha}{2}\varphi \, , \tag{C.2}$$

where the function $W(R)$ is defined as

$$W(R) = \left(\frac{f_1}{f_0} \right)^{1/4} \, , \quad f_0 = 1 + \frac{R^2}{4}, \quad f_1 = 1 + \frac{R^2}{4}\alpha^2 \, . \tag{C.3}$$

Then it is straightforward to verify that the solution (3.14), (3.15), (3.16) is transformed to the one of [9]

$$ds_6^2 = \rho_+ \left\{ W^2\eta_{\mu\nu}dx^\mu dx^\nu + R_0^2 L^2 \left[dR^2 + c_0^2 A^2 d\psi^2 \right] \right\} \, , \tag{C.4}$$

$$\mathcal{F}_{R\psi} = M^2 c_0 R_0 \alpha \cdot \frac{L^2 A}{W^6} \, , \tag{C.5}$$

$$\chi = M^2 \ln(\rho_+ W^2) \, , \tag{C.6}$$

with
$$L(R) = \frac{W}{f_0} \quad \text{and} \quad A(R) = \frac{R}{W^4} \ . \tag{C.7}$$

Note that here we keep c_0, although in [9] a choice of $c_0 = 1$ was made. The range of the angular coordinate has been altered by this transformation, as $\psi \in \left[0, 2\pi\xi\frac{1+\alpha}{2}\right)$. To compare with [9], where $\psi \in [0, 2\pi)$, one should make the choice $\xi = 2/(1 + \alpha)$. Then the deficit angles are given by $\beta_+ = c_0$ and $\beta_- = c_0/\alpha^2$. Note that in the conventions of [9] there is $\alpha = q/(4g)$, with q the magnetic charge and g the gauge coupling.

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